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On approximations of trigonometric Fourier series as transformed to alternating series

Zbigniew Płochocki*, Andrew Mioduchowski**
*Institute of Fundamental Technological Research, ul.Pawińskiego 5B, 02-106 Warsaw, Poland,
** University of Alberta, Dept. of Mechanical Engineering, Edmonton, Alberta, Canada T6G 2G8

Abstract. Trigonometric Fourier series are transformed into: (1) alternating series with terms of finite sums or (2) finite sums of alternating series. Alternating series may be approximated by finite sums more accurately as compared to their simple truncation. The accuracy of these approximations is discussed.
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I. Introduction

Trigonometric Fourier series have many applications, among others in solving some problems with (partial) differential equations, especially concerning heat conduction problems (see for example [1], [2], [3]). In such cases one or two of the following series occur:

(1.1)
$$\left.\begin{aligned} F1(x) &= \sum_{k=1}^{\infty} a_k \cos \mu_k x \\ F2(x) &= \sum_{k=1}^{\infty} b_k \sin \mu_k x \end{aligned}\right\} \mu_k = k\pi , \qquad \left.\begin{aligned} F3(x) &= \sum_{k=1}^{\infty} \tilde{a}_k \cos \tilde{\mu}_k x \\ F4(x) &= \sum_{k=1}^{\infty} \tilde{b}_k \sin \tilde{\mu}_k x \end{aligned}\right\} \tilde{\mu}_k = (2k-1)\frac{\pi}{2}$$

where: (1) all the series are assumed to be convergent; (2) the coefficients a_k , $\tilde{a}_k$, b_k , $\tilde{b}_k$ represent suitable Fourier transforms of an analyzed function; (3) and the further analysis of the above series is limited to $x \in [0,1]$, since (3a) each function may be presented as a sum of its even and odd parts ($f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$), (3b) cosine is an even function and sine is an odd one, and (3c) cosine and sine are both periodical functions (in some cases this limitation follows immediately from the meaning of the considered problem).

In order to obtain detailed information from the Fourier series (representing the solution of a given problem), usually it is necessary to perform some numerical calculations. Two problems then arise: (1) approximation of real values of independent variable x by rational numbers, and (2) criteria of approximation of the series by finite sums within a given accuracy.

The solution to the first problem is known and used in all numerical calculations: each real number can be approximated with arbitrarily assumed accuracy by a rational number, i.e. by the ratio p/q, where p and q stand for integers. Therefore in order to determine values of the series given by Eqs.(1.1) within a given accuracy it is sufficient to determine their values for variable x belonging to a sufficiently dense set of rational numbers. In our case, it is therefore sufficient to determine values of the considered series for a suitable set of rational values of variable

(1.2)
$$x = \frac{p}{q} , \quad 0 \le p \le q \ne 0 .$$

Thus, in further course the following series will be considered (the argument $x = p/q$ will not be notified, for simplicity):

$$\text{(1.3)}\quad \left.\begin{aligned} F1 &= \sum_{k=1}^{\infty} a_k \cos\mu_k \frac{p}{q} \\ F2 &= \sum_{k=1}^{\infty} b_k \sin\mu_k \frac{p}{q} \end{aligned}\right\} \mu_k = k\pi\,, \qquad \left.\begin{aligned} F3 &= \sum_{k=1}^{\infty} \tilde{a}_k \cos\tilde{\mu}_k \frac{p}{q} \\ F4 &= \sum_{k=1}^{\infty} \tilde{b}_k \sin\tilde{\mu}_k \frac{p}{q} \end{aligned}\right\} \tilde{\mu}_k = (2k-1)\frac{\pi}{2}\,,$$

with p and q satisfying Ineq.(1.2).

The aim of this paper is to solve the second problem mentioned above for some series of the type (1.3), i.e. to find possible general approximations of such series by finite sums and accuracy criteria for these approximations.

II. Simple transformation

Each of the trigonometric functions in Eqs.(1.3) periodically change the sign and repeat its absolute value with increasing k, i.e. the following relationships take place:

$$\text{(2.1)}\quad \begin{aligned} &\cos(k+nq)\pi\frac{p}{q} = (-1)^{np}\cos k\pi\frac{p}{q}\,, \quad \cos[2(k+nq)-1]\frac{\pi}{2}\frac{p}{q} = (-1)^{np}\cos(2k-1)\frac{\pi}{2}\frac{p}{q}\,, \\ &\sin(k+nq)\pi\frac{p}{q} = (-1)^{np}\sin k\pi\frac{p}{q}\,, \quad \sin[2(k+nq)-1]\frac{\pi}{2}\frac{p}{q} = (-1)^{np}\sin(2k-1)\frac{\pi}{2}\frac{p}{q}\,. \end{aligned}$$

Using these relationships one may rewrite each series (1.3) in the two following forms:

$$\text{(2.2)}\quad \begin{aligned} F1 &= \sum_{n=0}^{\infty}(-1)^{np}\sum_{k=1}^{q} a_{k+nq}\cos\mu_k\frac{p}{q}\,, & F3 &= \sum_{n=0}^{\infty}(-1)^{np}\sum_{k=1}^{q} \tilde{a}_{k+nq}\cos\tilde{\mu}_k\frac{p}{q}\,, \\ &= \sum_{k=1}^{q}\cos\mu_k\frac{p}{q}\sum_{n=o}^{\infty}(-1)^{np}a_{k+nq}\,, & &= \sum_{k=1}^{q}\cos\tilde{\mu}_k\frac{p}{q}\sum_{n=o}^{\infty}(-1)^{np}\tilde{a}_{k+nq}\,, \\ F2 &= \sum_{n=0}^{\infty}(-1)^{np}\sum_{k=1}^{q} b_{k+nq}\sin\mu_k\frac{p}{q}\,, & F4 &= \sum_{n=0}^{\infty}(-1)^{np}\sum_{k=1}^{q} \tilde{b}_{k+nq}\sin\tilde{\mu}_k\frac{p}{q}\,, \\ &= \sum_{k=1}^{q}\sin\mu_k\frac{p}{q}\sum_{n=o}^{\infty}(-1)^{np}b_{k+nq}\,, & &= \sum_{k=1}^{q}\sin\tilde{\mu}_k\frac{p}{q}\sum_{n=o}^{\infty}(-1)^{np}\tilde{b}_{k+nq}\,, \end{aligned}$$

The first form does not require any additional assumptions on the considered series, since the series is summated term by term without changes of summation order. The second form does not change the value of the series if it is absolutely converging, because summation breaks the order of summation of the initial series.

However the truncated second form is always equal to the truncated first one (at the same level), because final sum is independent of the order of summation. Additionally, criteria for treating the first form as an alternating series are more complicated and generally difficult to satisfy (see Remark 1 in Sec.4), whereas the second form has simpler structure and is more convenient in applications. For this reason the further analysis will be limited to the second form only.

The second form of (2.2) allows the initial series to be approximated by a finite sum of suitable approximated alternating series. This approximation will be discussed in Sec.4.

III. On another note

The second form of (2.2) may be especially useful for series of the type

$$\text{(3.1)}\quad S = \sum_{n=0}^{\infty} A_n\,, \quad |A_{n+1}| \le \alpha |A_n|\,, \quad 0 < \alpha < 1\,.$$

The absolute error Δ_N of truncation of such a series on level N (the absolute value of the truncation remainder $R_N = |S - S_N|$, $S_N = \sum_{n=0}^{N} A_n$) may be estimated as (see [4], [5]) (3.2) $$\Delta_N = R_N \leq \frac{\alpha}{1-\alpha} | A_N | \quad ,$$

and relative truncation error as

(3.3) $$\delta_N^\alpha = \frac{\Delta_N}{|S|} \cong \frac{\Delta_N}{|S_N|} \leq \frac{\alpha}{1-\alpha} \frac{| A_N |}{|S_N|}$$

(approximated equality holds if R_N is sufficiently small as compared to S_N).

IV. Towards alternating series

The second form of (2.2) represents alternating series (see Appendix), if (sufficient conditions at a given p and q):

– p is an odd integer and the terms Al_{kn} (with

(4.1) $$A1_{kn} = a_{k+nq} \ , \qquad A3_{kn} = \tilde{a}_{k+nq} \ ,$$
$$A2_{kn} = b_{k+nq} \ , \qquad A4_{kn} = \tilde{b}_{k+nq} \quad)$$

constitute a positive decreasing sequences (with respect to n):

(4.2) $$Al_{kn} > Al_{k(n+1)} > 0 \ , \quad l = 1, \ 2, \ 3, \ 4 \ ,$$

respectively;

– or p is an even integer and the terms Al_{kn} constitute alternating decreasing sequences (with respect to n):

(4.3) $$Al_{kn} = (-1)^n \, | Al_{kn} | \ , \quad | Al_{kn} | > | Al_{k(n+1)} | \ , \quad l = 1, \ 2, \ 3, \ 4 \ ,$$

respectively.

Remark 1: Analogous conditions may be formulated for the first form of (2.2). Then it will be seen that such conditions depend on the structure of the considered series and values of p and q, and it is not always possible to satisfy them, whereas conditions for the second form of (2.2) are independent of p and q.

In conclusion: If the above conditions are satisfied, then the initial series of (1.3) may be transformed into a finite sum of an alternating series.

V. Approximation of Fourier series as transformed into alternating series

Usually, numerical information from a given series is obtained through approximation by a suitable finite sum using the truncated series. In our case these truncated series are given by the second form of (2.2) with the second sums limited to truncation level N instead of ∞.

Remark 2: These truncated series are equal to initial ones (1.3) truncated at level K (after the K-th term), i.e. with the sums limited to K instead of ∞, with

(5.1) $$K = (N+1) \ q \ .$$

Series (2.2) may be treated as alternating series (see Appendix), if the criteria mentioned above [see Eqs. (4.2)] are satisfied. In the simple approach such series may be estimated by finite sums F_N and F_{N-1} being the alternating series truncated after the N-th and the $(N-1)$-th terms, respectively (see Appendix). However, a more accurate approximation of such series is proposed here (see Appendix). Applying this proposal the series occurring in (2.2) [or in compact form in (2.2') (see footnote [4])] may be approximated as follows:

(5.2)
$$F1 \cong \overline{F}1 = \sum_{k=1}^{q} \cos \mu_k \frac{p}{q}\left(\sum_{n=0}^{N} (-1)^n a_{k+nq} - (-1)^N \frac{1}{4} a_{k+Nq} \right) ,$$
$$F2 \cong \overline{F}2 = \sum_{k=1}^{q} \sin \mu_k \frac{p}{q}\left(\sum_{n=0}^{N} (-1)^n b_{k+nq} - (-1)^N \frac{1}{4} b_{k+Nq} \right) ,$$
$$F3 \cong \overline{F}3 = \sum_{k=1}^{q} \cos \tilde{\mu}_k \frac{p}{q}\left(\sum_{n=0}^{N} (-1)^n \tilde{a}_{k+nq} - (-1)^N \frac{1}{4} \tilde{a}_{k+Nq} \right) ,$$
$$F4 \cong \overline{F}4 = \sum_{k=1}^{q} \sin \tilde{\mu}_k \frac{p}{q}\left(\sum_{n=0}^{N} (-1)^n \tilde{b}_{k+nq} - (-1)^N \frac{1}{4} \tilde{b}_{k+Nq} \right) ,$$

where μ_k and $\tilde{\mu}_k$ are given by Eqs.(1.1).

The (relative) errors of these approximations (as referred to $|F_N|$) may be generally estimated as follows:

(5.3)
$$\overline{\delta}1_N \le \frac{1}{4|F_N|} \sum_{k=1}^{q} \left| \cos \mu_k \frac{p}{q} a_{k+Nq} \right| , \quad \overline{\delta}3_N \le \frac{1}{4|F_N|} \sum_{k=1}^{q} \left| \cos \tilde{\mu}_k \frac{p}{q} \tilde{a}_{k+Nq} \right| ,$$
$$\overline{\delta}2_N \le \frac{1}{4|F_N|} \sum_{k=1}^{q} \left| \sin \mu_k \frac{p}{q} b_{k+Nq} \right| , \quad \overline{\delta}4_N \le \frac{1}{4|F_N|} \sum_{k=1}^{q} \left| \sin \tilde{\mu}_k \frac{p}{q} \tilde{b}_{k+Nq} \right| ,$$

respectively. These errors are four times smaller than the estimated general truncation errors.

Remark 3: See Remark A2 in Appendix.

Remark 4: Of course, level N of a given approximation should be determined according to an assumed accuracy of approximation.

VI. Sample computing

As an example we present a few results of numerical analysis to the series:

(6.1)
$$F = \sum_{k=1}^{\infty} \cos k\pi \frac{p}{q} \exp[-k^2 \pi^2 t] .$$

After the simple transformation presented in Sec.2 with p assumed to be an odd number it may be rewritten in the form of the following finite sum of alternating series [see Eq.(2.2)$_1$]:

(6.2)
$$F = \sum_{k=1}^{q} \cos k\pi \frac{p}{q}\left(\sum_{n=0}^{\infty} (-1)^n \exp[-(k+nq)^2 \pi^2 t] \right)$$

(the second form is adopted as more convenient for analysis).

Remark 5: **W**e will use truncated series, therefore the requirement of absolutely converging of series (6.1) to be transformed into series (6.2) is not important (final sum is independent of the summation order), although series (6.1) is absolutely converging.

First of all we shall illustrate the accuracy of approximation (5.2) for $t = 0.01$, 0.05 , 0.1 and $p/q = 1/10$, $1/4$, $1/2$, $3/4$, $9/10$ (with possible cases of mp/mq). We assume that the estimated relative error of this approximation does not exceed $O^* = 5 \cdot 10^{-5}$ (this corresponds to the accuracy of numerical evaluation in round to 5 digits). In each particular case level N of the approximation satisfying the assumed accuracy is found [and corresponding level $K = (N+1)\ q$] with corresponding estimated error $\overline{\delta}_N$. For comparison there were computed also: the exact values of series (6.1), the values of truncated series F_N [and $F_K = F_N$ with $K = (N+1)\ q$], estimated error $\delta_N = |(F - F_N)/F_N|$ [$= \delta_K = |(F - F_K)/F_K|$, with $K = (N+1)\ q$].

In addition, each alternating series (6.2) is of the type (3.1) with

(6.3) $$0 < \alpha = \alpha_k \le \exp[-(q^2 + 2qk)\pi^2 t] < \alpha_{\max} = \exp[-(q^2 + 2q)\pi^2 t]$$

[the ratio of the $(n+1)$ -th term to the n -th is the greatest one for $n = 0$]. Therefore the (relative) error of truncation of series (6.2) on level N may be estimated according to (3.3) as

(6.4) $$\delta_N^\alpha \le \frac{1}{|F_N|}\sum_{k=1}^{q}\left|\cos k\pi \frac{p}{q}\right| \frac{\alpha_k}{1-\alpha_k}\exp[-(k+Nq)^2\pi^2 t]$$

The values of these estimated errors are also included in the Tables, with the values of $\alpha_{\max}$ for orientation.

All the results of these calculations are given in Tables 1, 2 and 3. Suitable numbers are given in round to 5 digit (although calculations were performed with suitable higher accuracy, in some cases in round even to 200 digits, if it was necessary).

Table 1.
Table 2.
Table 3.

Appendix. On alternating series and their approximations

Alternating series (S) is understood as a (convergent) series of the type:

(A.1) $$S = \sum_{n=0}^{\infty}(-1)^n c_n \;; \quad c_n > c_{n+1} > 0 \;,\; n = 0,\ 1,\ 2,\ \ldots \;;\; \lim_{n\to\infty} c_n = 0 \;;$$

the latter relationship is the condition for convergence of the series (according to the Leibniz criterion).

Remark A1: A little more general (converging) series with the condition $c_n > c_{n+1} > 0$, $n = n_0,\ n_0+1,\ n_0+2,\ \ldots$ instead of (A.1) $_2$ may be treated as the sum of a finite sum and an alternating series as defined above.

If S is truncated at an even level $N = 2j$ (i.e. after the N -th term) then it may be estimated as follows. Since $S = S_{2j} - \sum_{n=2j+1}^{\infty}(-1)^{n+1}c_n < S_{2j}$ and $S = S_{2j-1} + \sum_{n=2j}^{\infty}(-1)^n c_n > S_{2j-1}$, therefore

(A.2a) $$S_{N=2j-1} \le S \le S_{N=2j} \,,$$

where

(A.3) $$S_N = \sum_{n=0}^{N}(-1)^n c_n \;;$$

if S is truncated at an odd level $N = 2j+1$ then $S = S_{2j} - \sum_{n=2j+1}^{\infty}(-1)^{n+1}c_n < S_{2j}$ and $S = S_{2j+1} + \sum_{n=2j+2}^{\infty}(-1)^n c_n > S_{2j+1}$, therefore

(A.2b) $$S_{N=2j+1} \le S \le S_{N=2j} \,.$$

Since

(A.4) $$S_N - S_{N-1} = (-1)^N c_N \,,$$

therefore the absolute error of truncation of S at level N cannot exceed the last term c_N taken into account (the last term of S_N). The relative error of such a truncation (as referred to the smaller S_N , i.e, to $S_{N\min} = S_N - \frac{1}{2}[1 + (-1)^N]\ c_N$) may be estimated as follows:

(A.5) $$\hat{\delta}_N \le \frac{c_N}{S_{N\min}} \,.$$

However, S may be approximated a little more exactly (and therefore also the approximation error may be estimated a little more precisely as compared to the truncation error). In order to find such a general approximation we divide S as follows:

(A.6) $$S = S_N - (-1)^N R_N \ , \quad R_N = (-1)^{N+1} \sum_{n=N+1}^{\infty} (-1)^n c_n > 0 \, ,$$

or in details (see Fig.A1.)

(A.6') $$S = S_{2j} - R_{2j} \ , \qquad R_{2j} = c_{2j+1} - c_{2j+2} + \ \dots \quad ,$$
$$= S_{2j+1} + R_{2j+1} \ , \quad R_{2j+1} = c_{2j+2} - c_{2j+3} + \ \dots \, .$$

Fig.A.1

Since of (A.1) $_{2,3}$ the differences of the two neighboring terms of S constitute a positive monotonically decreasing sequence, and therefore generally

(A.7) $$R_N < R_{N-1}$$

for each S. Taking into account (A.2) and (A.4) one may conclude that to obtain better approximation of S (as compared to S_N) it is sufficient to add/subtract a fraction of c_N to/from S_N in the case of odd/even N, respectively (see Fig.A.1). Thus, one has to consider the following approximation:

(A.8) $$S \cong S_N^x = S_N - (-1)^N x c_N = S_{N-1} + (1-x)(-1)^N c_N \ , \quad 0 < x < 1 \, .$$

The (absolute) error of such an approximation is

(A.9) $$\Delta_N^x = | S - S_N^x | = | R_{N-1} - (1-x) c_N | \le x c_N \, ,$$

where (A.8) $_2$, (A.6) with (A.4), and the estimation

(A.10) $$R_{N-1} = c_N - \sum_{m=1}^{\infty} (-1)^{m+1} c_{N+m} \le c_N$$

are used. Since of (A.7) and (A.4) the following relationships take place:

(A.11) $$\frac{1}{2} \le \frac{R_{N-1}}{c_N} \le 1$$

and therefore the quantity

(A.12) $$\frac{\Delta_N^x}{c_N} = \left| \frac{R_{N-1}}{c_N} - (1-x) \right|$$

has the lowest upper limitation in the whole domain of ${R_{N-1}}/{c_N}$ [(see (A.11)] for $x = 1/4$ (see Fig.A.2).

Fig.A.2

Thus, the best general approximation of S is

(A.13) $$S \cong \overline{S}_N = S_N - (-1)^N \frac{1}{4} c_N = S_{N-1} + (-1)^N \frac{3}{4} c_N \, ,$$

or in details

(A.13') $$S \cong \overline{S}_{2j} = S_{2j} - \frac{1}{4} c_{2j} = S_{2j-1} + \frac{3}{4} c_{2j} \ , \qquad N = 2j \ ,$$
$$S \cong \overline{S}_{2j+1} = S_{2j+1} + \frac{1}{4} c_{2j+1} = S_{2j} - \frac{3}{4} c_{2j+1} \ , \quad N = 2j+1 \, .$$

The (absolute) error of this approximation

(A.14) $$\overline{\Delta}_N = | S - \overline{S}_N | = R_{N-1} - \frac{3}{4} c_N < c_N - \frac{3}{4} c_N = \frac{1}{4} c_N$$

[where the estimation (A.10) is used] does not exceed ¼ of the last term taken into account. The relative error (as referred to the smaller S_N , i.e, to $S_{N\min} = S_N - \frac{1}{2}[1+(-1)^N]\ c_N$)

(A.15) $$\overline{\delta}_N \le \frac{1}{4}\frac{c_N}{S_{N\min}}$$

is four times smaller than the estimated truncation error [cf. Ineq.(A.5)].

Remark A2: In particular case of alternating series of type (3.1) the truncation at the level N may generate smaller estimated error (3.2) as compared to (A.15), if $\alpha < 1/2$ [but if approximation (A.13) is applied instead of a truncation , then the estimated approximation error is smaller if $\alpha < 1/5$!] .

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Table 1. $t = 0.01$

p/q	F *)	N	$\overline{\delta}_N$	K	δ_K **)	δ_N^α	$\alpha_{\max}$
$1/10$	1.6970	1	$9.98\cdot10^{-7}$	20	$7.10\cdot10^{-20}$	$2.65\cdot10^{-11}$	$7.18\cdot10^{-6}$
$1/4$	−0.49455	2	$6.68\cdot10^{-4}$				
		3	$1.11\cdot10^{-7}$	16	$3.17\cdot10^{-12}$	$4.56\cdot10^{-8}$	0.0936
$3/12$		1	$1.11\cdot10^{-7}$	24	$1.26\cdot10^{-26}$	$2.78\cdot10^{-14}$	$6.29\cdot10^{-8}$
$1/2$	0.091303	3	$9.13\cdot10^{-4}$				
		4	$2.61\cdot10^{-5}$	10	$1.35\cdot10^{-6}$	$4.61\cdot10^{-5}$	0.454 ***)
$3/6$		1	$9.40\cdot10^{-4}$				
		2	$2.01\cdot10^{-9}$	18	$1.45\cdot10^{-17}$	$2.16\cdot10^{-11}$	0.00876
$5/10$		1	$3.42\cdot10^{-7}$	20	$3.63\cdot10^{-21}$	$1.36\cdot10^{-12}$	$7.18\cdot10^{-6}$
$3/4$	−0.499998	2	$1.22\cdot10^{-4}$				
		3	$2.02\cdot10^{-8}$	16	$5.79\cdot10^{-13}$	$8.34\cdot10^{-9}$	0.0936
$9/12$		1	$2.03\cdot10^{-8}$	24	$2.30\cdot10^{-27}$	$5.08\cdot10^{-15}$	$6.29\cdot10^{-8}$
$9/10$	−0.500000	1	$3.38\cdot10^{-6}$	20	$2.35\cdot10^{-19}$	$9.00\cdot10^{-11}$	$7.18\cdot10^{-6}$

*) Also F_N [$= F_K$ with $K = (N+1)\ q$] and $\overline{F}_N$, since all these quantities are equal within an accuracy of $O^* = 5.10^{-5}$. For example: for $p/q = 1/4$

F = – 0.494554289 ,

F_4 = – 0.494554958 ,

$\overline{F}_4$ = – 0.494542027 .

**) $\delta_K = \delta_N$ with $K = (N+1)\ q$.

***) Note that in this case $\alpha_{\max} > 1/5$ (see Remark A2).

Table 2. $t = 0.05$

p/q	F *)	N	$\overline{\delta}_N$	K	δ_K **)	δ_N^α	$\alpha_{\max}$
$1/10$	0.70004	1	$3.97 \cdot 10^{-27}$	20	$4.17 \cdot 10^{-95}$	$3.04 \cdot 10^{-52}$	$1.91 \cdot 10^{-26}$
$1/4$	0.42298	1	$1.83 \cdot 10^{-6}$	8	$7.30 \cdot 10^{-18}$	$5.27 \cdot 10^{-11}$	$7.18 \cdot 10^{-6}$
$1/2$	–0.13854	1	$6.72 \cdot 10^{-4}$				
		2	$3.48 \cdot 10^{-8}$	6	$1.39 \cdot 10^{-13}$	$3.74 \cdot 10^{-10}$	0.0193
$3/6$		1	$3.47 \cdot 10^{-14}$	12	$7.12 \cdot 10^{-42}$	$1.92 \cdot 10^{-26}$	$5.16 \cdot 10^{-11}$
$3/4$	–0.42373	1	$1.83 \cdot 10^{-6}$	8	$7.29 \cdot 10^{-18}$	$5.26 \cdot 10^{-11}$	$7.18 \cdot 10^{-6}$
$9/10$	–0.47505	1	$5.85 \cdot 10^{-27}$	20	$6.14 \cdot 10^{-95}$	$4.48 \cdot 10^{-52}$	$1.91 \cdot 10^{-26}$

*) Also F_K [$= F_N$ with $K = (N+1)\ q$] and $\overline{F}_N$, since all these quantities are equal in round to 5 digits. For example for $p/q = 1/2$

F = – 0.138538850957

F_2 = – 0.138538850958

$\overline{F}_2$ = – 0.138538802810 .

**) $\delta_K = \delta_N$ with $K = (N+1)\ q$.

Table 3. $t = 0.1$

p/q	F *)	N	$\overline{\delta}_N$	K	δ_K **)	δ_N^α	$\alpha_{\max}$
$1/10$	0.37016	1	$8.78 \cdot 10^{-53}$	20	$2.42 \cdot 10^{-189}$	$1.29 \cdot 10^{-103}$	$3.67 \cdot 10^{-52}$
$1/4$	0.26345	1	$1.29 \cdot 10^{-11}$	8	$5.12 \cdot 10^{-35}$	$2.67 \cdot 10^{-21}$	$5.16 \cdot 10^{-11}$
$1/2$	–0.019296	1	$1.80 \cdot 10^{-6}$	4	$1.92 \cdot 10^{-14}$	$5.16 \cdot 10^{-11}$	$3.72 \cdot 10^{-4}$
$3/4$	–0.26345	1	$1.29 \cdot 10^{-11}$	8	$5.12 \cdot 10^{-35}$	$2.67 \cdot 10^{-21}$	$5.16 \cdot 10^{-11}$
$9/10$	–0.33894	1	$9.59 \cdot 10^{-53}$	20	$2.64 \cdot 10^{-189}$	$1.41 \cdot 10^{-103}$	$3.67 \cdot 10^{-52}$

*) Also F_K [$= F_N$ with $K = (N+1)\ q$] and $\overline{F}_N$, since all these quantities are equal in round to 5 digits. For example for $p/q = 1/2$

F = – 0.0192961642685008367 4 ,

$F_1 = \overline{F}_1$ = – 0.019296164268500 46583 .

**) $\delta_K = \delta_N$ with $K = (N+1)\ q$.

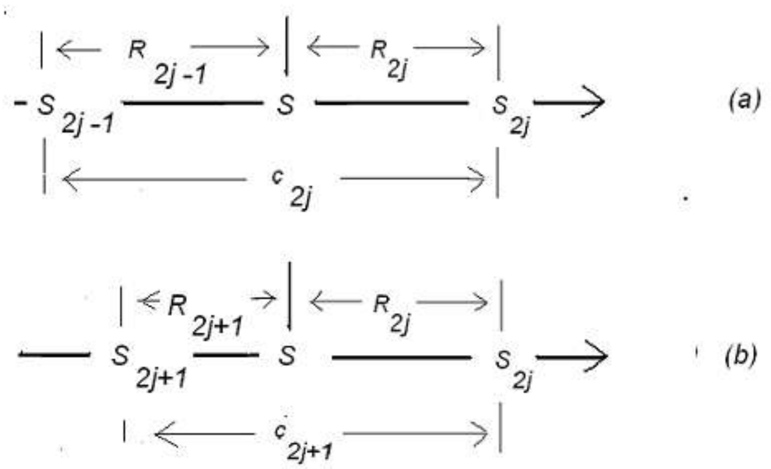

Fig. A1. Interpretation of (A.3) for $N = 2j$ (a) and $N = 2j + 1$ (b) [see (A.6')]

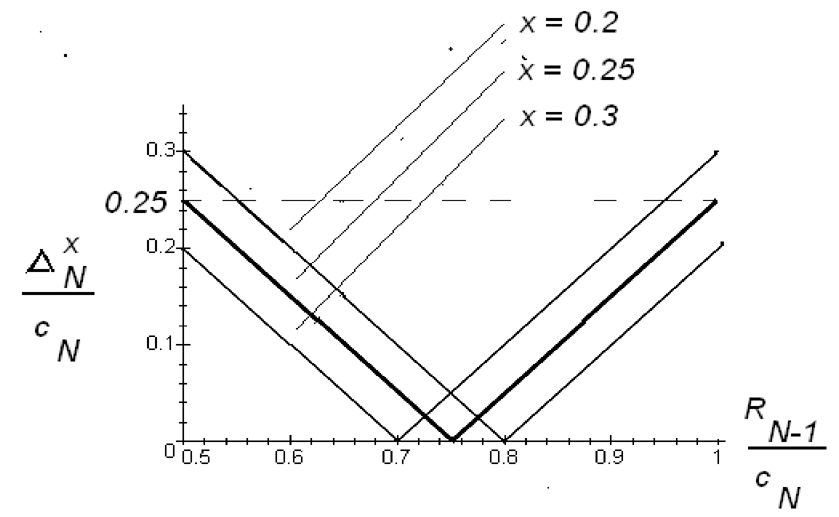

Fig.A.2. Optimization of error Δ_N^x